Counterexample of a Claim Pertaining to the Synthesis of a Recurrent Neural Network

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Abstract - Recurrent Neural Networks has received much attention due to its nonlinear dynamic behavior. One such type of dynamic behavior is that of setting to a fixed stable state. This paper will show that a counterexample to the synthesis procedure [22] for sparsely interconnected recurrent neural networks.

I. INTRODUCTION

Neural Networks with feedback connections attract considerable attentions for a number of reasons. Biological neural networks are highly recurrently connected. From engineering viewpoint, time-varying behaviors are probably of greater importance among the nonlinear dynamic behaviors that recurrent neural networks manifest. Many authors have studied recurrent neural network models of various types of perceptual processes and applications [1], [2], [3], [4], [5], [6], [7].

One of the dynamic behaviors concerned by many works [8], [9], [10], [11], [12] is that the existence and the location of equilibrium points, and with the qualitative properties of the equilibria. The stability analysis and applications of a class of single layered, fully or sparsely connected neural networks have also been of recent interest [13], [14], [15]. In [16], [17], [18], [19], several synthesis procedures are developed for different types of neural networks models (see [20] for an overview of some of these procedures). Among these synthesis procedures, the *eigenstructure method* [9], [10], [13], [18] appears to be especially effective. In the present paper, however, we provide a counterexample of this synthesis procedure based on Winner-Take-All network.

In Section II, we will define the form of recurrent neural networks. Some analysis and proofs are presented in Section III, and our conclusion comes in Section IV

II. NETWORK ARCHITECTURE

Since the beginning of neural networks research, the Winter-Take-All network has played a very important role in the design of learning algorithms, in particular, most of unsupervised learning algorithms such as competitive learning, self organizing map and Adaptive Resonance Theory model; and has become the fundamental building block of many complex systems. So we choose Winner-Take-All as the test bench model.

The class of recurrent neural networks (RNN) (see figure 1) considered in this paper is defined of the form

$$\dot{x}(t) = -Ax(t) + TS(x(t)) + B \qquad (1)$$

where the input vector is x(0), with $x \in R^n$, is a real n-vector, $\dot{x}(t)$ denotes the time derivative of x, A is a real n x n diagonal matrix with positive elements, T is a real n x n matrix, B is a real n-vector of constant bias values, and the real n-vector valued function S(x) can be one of the following two forms:

(1) Each component $s_i(x_i)$ of S(x) is a *sigmoidal function*, which maps the real numbers *R* into real interval (-1, 1). It's smooth and monotonically increasing. A typical one is as follows:

$$s_i(x_i) = \tanh(x_i) = \frac{e^{x_i} - e^{-x_i}}{e^{x_i} + e^{-x_i}}, \text{ or}$$

(2) Each component of S(x) is a *saturation function* defined by

$$S(x_i) = \begin{cases} 1, & x_i > 1 \\ x_i, & -1 < x_i < 1 \\ -1, & x_i > 1. \end{cases}$$

Sometimes, the system (1) can be discretized, hence the system has discrete update equation

$$x(k+1) = S(Tx(k) + I) \qquad (2)$$

Note that the decay term A has been eliminated. We will focus mainly on this model in the paper.

In many applications, system (1) with sigmoidal function and symmetric T is often referred as *Hopfield model* [16]. Among other applications, system (1), with saturation function, has been used to store bipolar memories and as cellular neural networks [13], [17].

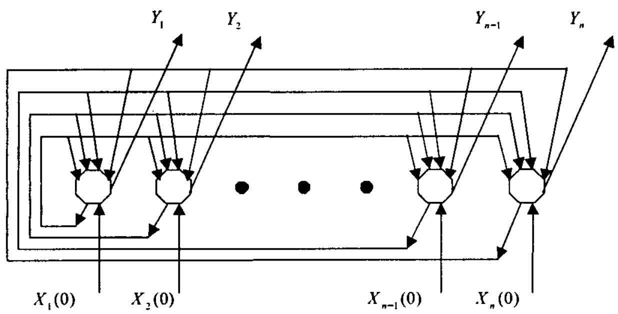

Figure 1: Architecture of recurrent neural networks

III. SOME PROOFS AND ANALYSIS

Let $P^n = \{x \in Z^n : x_i = 1 \text{ or } -1,\ i = 1,\ldots,n\}$ and $C(\alpha) = \{x \in R^n : x_i\alpha_i > 1, i = 1,\ldots,n\}$ for $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T \in P^n$. In an N-neurons Winner-Take-All case, $\alpha^k = [\alpha_1^k, \alpha_2^k, \ldots, \alpha_n^k]^T \in P^n$, where $\alpha_k^k = 1,\ \alpha_j^k = -1,\ \forall j \neq k,\ k = 1,\ldots,n$, is the desired output. If we define the T matrix in system (2) as

$$T_{ij} = \begin{cases} 1, & i = j \\ -\varepsilon, & i \neq j \end{cases} \qquad 0 < \varepsilon < 1/n, \qquad 1 \le i, j \le n,$$

system (2) becomes MAXNET [1]. We will mainly analysis the performance of MAXNET with saturation function.

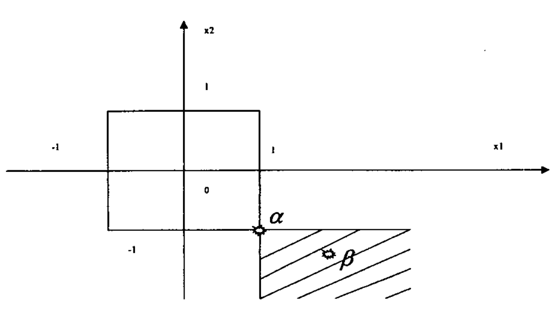

Figure 2. A geometric interpretation of stable equilibrium

Due to the properties of saturation function, the stable equilibrium of system (2), given by β, must be in the region of $C(\alpha)$ (see figure 2).

Proposition 1: If B is a zero vector and n>3, then the MAXNET can not achieve the desired output $\alpha^k \in P^n$.

Proof: If the system (2) can achieve α^k in finite time, then, at the equilibrium point, the system (2) has

$$\alpha^k = \mathrm{S}(\mathrm{T}\alpha^k) \text{ or } \mathrm{T}\alpha^k \in C(\alpha^k) \tag{3}$$

which can be equivalently written as

$$1+(n-1)\varepsilon > 1 \text{ when } \alpha_i^k = 1; \tag{4}$$

and

$$-1+(n-3)\varepsilon < -1 \text{ When } \alpha_i^k = -1 \tag{5}$$

From (4), we have ε >0, and (5), ε <0. It's a contradiction. □

Note that MAXNET with other type of piecewise nonlinear function may achieve the desired output α^k [21].

Proposition 2: If B is a non-zero vector, then the MAXNET can achieve the desired output $\alpha^k \in P^n$.

Proof: similarly as proposition 1, we have

$$\alpha^k = \mathrm{S}(\mathrm{T}\alpha^k + \mathrm{B}) \text{ or } \mathrm{T}\alpha^k + \mathrm{B} \in C(\alpha^k) \tag{6}$$

at the equilibrium point, and hence we have

$$-(n-1)\varepsilon < B_i < -(n-3)\varepsilon \tag{7}$$

As 0< ε <1/n and n is finite, there exists a non-empty interval, though very small, for B_i such that (6) holds. □

Note that the convergence rate of MAXNET will decrease as n increases and the initial inputs have nearly the same values.

For system (2) with predefined constraints, such as some elements, not the diagonal ones, in T matrix are set to zeros. We are given an index matrix $D = [D_{ij}]$ with $D_{ii} \neq 0$ for i=1,…,n. Denote the i^{th} row of the index matrix D by $D_i = [D_{i1}, \ldots, D_{in}]$ [13]. For each i=1,…,n constructs two sets M_i and N_i, such that $M_i \cup N_i = \{1,\ldots,n\}$ and $M_i \cap N_i = \phi$, and D_{ij}=1 if $j \in M_i$, D_{ij}=0 if $j \in N_i$.

The restriction of matrix T to an index matrix D, denoted by T|D, is defined by T|D=[W_{ij}], where

$$W_{ij} = \begin{cases} T_{ij}, & if\ D_{ij} = 1 \\ 0, & otherwise \end{cases}.$$

In [22], Michel et al. states that "Sparsely constraints on the interconnecting structure for a given neural network are usually expressed as constraints which require that predetermined elements of T be zero. To simplify the subsequent discussion, we consider without any loss of generality the specific case when n=4 and the constrains on T are given, for example, by

$$T = \begin{bmatrix} T_{11} & 0 & T_{13} & 0 \\ 0 & T_{22} & 0 & T_{24} \\ T_{31} & 0 & T_{33} & 0 \\ 0 & T_{42} & 0 & T_{44} \end{bmatrix} \quad (25)$$

where the T_{ij}'s are to be determined. The question to be answered is weather for a given 4 x (r-1) matrix Y, it is possible to determine (non-trivial) solutions of T with structure (25) from the matrix equation (19), i.e. $TY = \mu Y$. We will show in the following that (non-trivial) solutions for such T always exists as long as all the diagonal elements of matrix T are assumed to be non-prespecified elements (e.g., as given in (25)) and $p<n$ (p is defined in (22), i.e. $p=rank(Y)$)." In the following proposition, however, we will show that such non-trivial solution does not exist in Winner-Take-All network.

Proposition 3: system (2) with restriction of matrix T can not achieve the desired output $\alpha^k \in P^n$.

Proof: In such a case, (6) can be written as

$$1- \sum_{j\in M_i} T_{ij} + B_i > 1 \text{ when } \alpha_i^k = 1; \quad (8)$$

$$-1- \sum_{j\in M_i, j\neq k} T_{ij} + T_{ik} + B_i < -1$$

$$\text{When } \alpha_i^k = -1 \text{ and } \alpha_k^k = 1 \text{ where } k \in M_i \quad (9)$$

$$-1- \sum_{j\in M_i} T_{ij} + B_i < -1$$

$$\text{When } \alpha_i^k = -1 \text{ and } \alpha_k^k = 1 \text{ where } k \in N_i \quad (10)$$

Clearly, we have $B_i > \sum_{j\in M_i} T_{ij}$ from (8) and $B_i < \sum_{j\in M_i} T_{ij}$ from (10). It's a contradiction.

System (2) with specified predefined constraints does not converge to the desired output. □

IV. CONCLUSION

In this paper, we have derived some properties of a class of recurrent neural networks. These properties provide an alternative approach to understand the dynamic behaviors of recurrent neural networks, both fully and sparsely interconnected ones.

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